Modular Almost Orthogonal Quantum Logics

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Almost orthogonal quantum logics, i.e., atomic orthomodular lattices in which to every atom there exist only finitely many nonorthogonal atoms, are studied. It is shown that an almost orthogonal quantum logic is modular if and only if it has the exchange property if and only if it can be embedded into a direct product of finite modular quantum logics. The class of almost orthogonal modular OMLs is the largest subclass of the class of atomic modular OMLs in which the conditions commutator-finite and block-finite are equivalent. A finite faithful valuation on an almost orthogonal quantum logic L exists if and only if L is modular and the set of all atoms of L is at most countable.

1. BASIC DEFINITIONS AND FACTS

By a *quantum logic* we mean an *orthomodular lattice* (OML) [see Kalmbach (1983) for details]. Recall that a nonzero element $a \in L$ is an *atom* if $0 \leq b < a$ implies $b = 0$. An OML L is *atomic* if every nonzero element in L contains an atom. An atomic OML L is *almost orthogonal* (Pulmannová and Riečanová, 1990*b*) if for every atom a in L the set $\mathbb{B}_a = \{b \in L \mid b$ is an atom in L and $b \not\leq a'\}$ is finite. We say that the OML L is a *compact topological OML* (Pulmannová and Riečanová, 1990*b*; Riečanová, 1989, 1990, 1991; Choe and Greechie, to appear) if there exists a compact Hausdorff topology τ on L such that for any x_α, y_α, x, $y \in L$ ($\alpha \in \mathbb{A}$, $\mathbb{A}$ is a directed set)

$$x_\alpha \xrightarrow{\tau} x \quad \text{implies} \quad x'_\alpha \xrightarrow{\tau} x'$$

$$x_\alpha \xrightarrow{\tau} x,\ y_\alpha \xrightarrow{\tau} y \quad \text{implies} \quad x_\alpha \vee y_\alpha \xrightarrow{\tau} x \vee y,\ x_\alpha \wedge y_\alpha \xrightarrow{\tau} x \wedge y$$

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A compact topological OML is *profinite* if it is a projective limit of finite OMLs. It was shown in Choe and Greechie (to appear) that an OML L is profinite if and only if it is a direct product of finite OMLs with their discrete topologies.

If L_1, L_2 are OMLs, then the mapping $\varphi: L_1 \to L_2$ is called a *morphism* if for all $x, y \in L_1$ the following hold: $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$, $\varphi(x') = (\varphi(x))'$, and $\varphi(0) = 0$. If φ is bijective and a morphism, then φ is called an *isomorphism*. The mapping $\varphi: L_1 \to L_2$ is called an *embedding* if $\varphi: L_1 \to \varphi(L_1) \subseteq L_2$ is an isomorphism [we sometimes identify L_1 with $\varphi(L_1)$]. A *residually finite* OML is an OML which can be embedded into a product of finite OMLs (Choe and Greechie, to appear). A complete ortholattice $\tilde{L}$ is the *MacNeille completion* of an OML L if L [precisely $\varphi(L)$, $\varphi: L \to \tilde{L}$ is an embedding] is join-dense and meet-dense in $\tilde{L}$. If L is atomic, then evidently $\tilde{L}$ is a complete atomic ortholattice such that $\varphi(L)$ and $\tilde{L}$ have the same set of all atoms. Recall that $x_\alpha \to^{(o)} x$ means that there exist $u_\alpha, v_\alpha \in L$ such that $u_\alpha \le x_\alpha \le v_\alpha$ and $u_\alpha \nearrow x$, $v_\alpha \searrow x$. The strongest of all topologies τ such that $x_\alpha \to^{(o)} x$ implies $x_\alpha \to^{\tau} x$ is called the *order topology* τ_0 (Birkhoff, 1967; Sarymsakov *et al.*, 1983; Erné and Weck, 1980). An OML L is called *(o)-continuous* if $x_\alpha \nearrow x$, $y_\alpha \nearrow y$ implies $x_\alpha \wedge y_\alpha \nearrow x \wedge y$ for any $x_\alpha, x, y_\alpha, y \in L$. Let us recollect some results concerning compact topological OMLs.

Theorem 1.1. Let L be an atomic OML. The following statements are equivalent:

(i) L is (o)-continuous and the interval topology τ_i in L is Hausdorff.
(ii) L is almost orthogonal.
(iii) The MacNeille completion $\tilde{L}$ of L is a compact topological OML in which L [precisely $\varphi(L)$] is a topologically dense subset.
(iv) For any atom $a \in L$, $L = \langle 0, a' \rangle \cup \bigcup_{k=1}^{n} \langle p_k, 1 \rangle$ (dually $L = \langle a, 1 \rangle \cup \bigcup_{k=1}^{n} \langle 0, p_k' \rangle$), where p_k are atoms in L, $p_k \not\le a'$ $(k = 1, 2, \ldots, n)$.

Proof. (ii)$\Leftrightarrow$(iv) is evident.

(iv)$\Rightarrow$(i): Let $a \in L$ be an atom. Then $L = \langle 0, a' \rangle \cup \bigcup_{k=1}^{n} \langle p_k, 1 \rangle$, where p_k are atoms in L, $p_k \not\le a'$ $(k = 1, 2, \ldots, n)$. Hence $\langle 0, a' \rangle$, $\bigcup_{k=1}^{n} \langle p_k, 1 \rangle$ are two disjoint clopen sets in τ_i. If $x, y \in L$, $x \ne y$, then there exists an atom $a \in L$ such that $x \in \langle 0, a' \rangle$, $y \notin \langle 0, a' \rangle$ (or $x \notin \langle 0, a' \rangle$, $y \in \langle 0, a' \rangle$). Thus τ_i is Hausdorff [see also Pulmannová and Riečanová (1990*b*), Theorem 2.3(i)].

Since $\tau_0 \supseteq \tau_i$, for any atom $a \in L$, the interval $\langle a, 1 \rangle$ is a clopen set in τ_0. Let $(x_\alpha)_{\alpha \in \mathbb{A}}$ be a net in L and let $x_\alpha \nearrow x$. Then $x_\alpha \to^{\tau_0} x$ and hence if $x \in \langle a, 1 \rangle$, $a \in L$ is an atom, there exists $\alpha_0 \in \mathbb{A}$ such that for every $\alpha \ge \alpha_0$ we have $x_\alpha \in \langle a, 1 \rangle$. This implies that for any $y \in L$, $y \wedge x_\alpha \nearrow y \wedge x$ [see Pulmannová and Rogalewicz (to appear), Proposition 6, for an alternative proof].

(i) $\Rightarrow$ (iii) follows by Riečanová (1991), Theorem 2.1.

(iii) $\Rightarrow$ (ii): Let the compact topological OML $(\tilde{L}, \tau)$ be a MacNeille completion of L. Since the lattice operations and the orthocomplementation are continuous in τ, the intervals $\langle a, 1\rangle$, $\langle 0, a'\rangle$ are clopen sets in τ for any atom $a \in \tilde{L}$. Since for every atom $a \in \tilde{L}$ we have $\tilde{L} = \langle 0, a'\rangle \cup \bigcup\{\langle p, 1\rangle \mid p \in L$ is an atom$\}$ and τ is compact, there exists a finite subcovering of $\tilde{L}$. Thus $\tilde{L} = \langle 0, a'\rangle \cup \bigcup_{k=1}^{n} \langle p_k, 1\rangle$, which implies that $\tilde{L}$, and hence also L, are almost orthogonal.

In particular, L is a compact topological OML if and only if L is complete and almost orthogonal. In a compact topological OML (L, τ) we have

$$x_\alpha \xrightarrow{(o)} x \text{ iff } x_\alpha \xrightarrow{\tau} x \text{ for any } x_\alpha, x \in L \text{ [abbreviated } \tau = (o)] \quad \blacksquare$$

2. THE EXCHANGE PROPERTY, MODULARITY, AND PROFINITENESS

The aim of this section is to show that for almost orthogonal OML the exchange property, modularity, and profiniteness (respectively residual finiteness) with modular factors are equivalent. We recall that an atomic OML has the *exchange property* if for every element x in L and all atoms a, b in L the following holds:

$$(b \wedge x = 0 \text{ and } b \leq a \vee x) \Rightarrow a \leq b \vee x$$

An OML L is *irreducible* if the center $C(L)$ of L is trivial, i.e., $C(L) = \{0, 1\}$. (Equivalently, L cannot be written in a nontrivial form of a direct product.)

Kalmbach (1983, p. 145, Theorem 12) states that in an atomic OML L the following conditions are equivalent:

(i) If a, b are distinct atoms in L, then there exists an atom c in $L \setminus \{a, b\}$ with $c \leq a \vee b$.

(ii) If A is finite orthogonal set of atoms in L and $|A| \geq 2$ holds, then there exists an atom c in L, $c \leq \bigvee A$ with not cCa for all $a \in A$ (cCa means that c is compatible with a).

Proposition 2.1. Any complete irreducible almost orthogonal OML L satisfying the exchange property is finite and modular.

Proof. Let $|L| > 2$. By Theorem 1.1, L is a compact topological OML. For every atom a in L the set

$$B_a = \{b \in L \mid b \text{ is an atom, } b \not\leq a'\}$$

is a finite and hence closed subset of L. Since $a \in B_a$, $B_a \neq \emptyset$ for every atom a in L. Assume that there exists an infinite set $A = \{a_1, a_2, \ldots\}$ of mutually orthogonal atoms in L. Since L is complete and irreducible and has the exchange property, every two atoms a, b in L are perspective, i.e., there is an atom $c \in L \setminus \{a, b\}$ with $c \leq a \vee b$ (Kalmbach, 1983, p. 142, Theorem 8). This implies that the above condition (i) and hence also (ii) are satisfied. In view of (ii), for every $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in A$ we have $\bigcap_{i=1}^{k} B_{a_i} \neq \emptyset$. The compactness of L then implies that $\bigcap_{i=1}^{\infty} B_{a_i} \neq \emptyset$, but this contradicts the almost orthogonality of L. Therefore, any set of mutually orthogonal atoms in L is finite, i.e., L has a finite height. By Example 16 of Kalmbach (1983, p. 156), L is modular. Let $\{a_1, a_2, \ldots, a_n\}$ be a maximal orthogonal set of atoms in L. Then for every atom $b \in L$, $b \neq a_i$ $(i = 1, 2, \ldots, n)$, we have $b \in B_{a_i}$ for some $i \in \{1, 2, \ldots, n\}$. Thus the set of all atoms in L is finite and hence L is finite. ∎

An alternative proof can be obtained taking into account that in an atomic OML with the exchange property the perspectivity of atoms is transitive and applying Pulmannová and Rogalewicz (to appear), Proposition 5.

Proposition 2.2. Every complete almost orthogonal OML L satisfying the exchange property is isomorphic to a direct product $\prod_{i \in I} [0, c_i]$, where c_i are atoms in $C(L)$ and $[0, c_i]$ $(i \in I)$ are finite irreducible modular OMLs.

Proof. Define a binary relation ρ on the set $A(L)$ of all atoms in L by putting $a\rho b$ iff $a \not\leq b'$. Let $\bar{\rho}$ be the transitive closure of ρ. By Pulmannová and Riečanová (1990*b*), if $\{T_i | i \in I\}$ is the set of all equivalence classes of $\bar{\rho}$, then for every $i \in I$ the element $c_i = \bigvee T_i$ is an atom in $C(L)$ and $\bigvee \{c_i | i \in I\} = 1$. Thus L is isomorphic to the direct product $\prod_{i \in I} [0, c_i]$. Moreover, $[0, c_i]$ are irreducible, complete almost orthogonal OMLs satisfying the exchange property. By Proposition 2.1, $[0, c_i]$ are finite modular OMLs. ∎

Remark 2.3. Let L be an almost orthogonal OML. If L has the exchange property, then the MacNeille completion $\tilde{L}$ of L has the exchange property. This follows from Maeda and Maeda (1970, p. 54, Theorem 12.7).

Theorem 2.4. Let L be an almost orthogonal OML. Then the following conditions are equivalent:

(i) L is modular.
(ii) L satisfies the exchange property.
(iii) The MacNeille completion of L is a direct product of finite irreducible modular OMLs.

Proof. (i) ⇒ (ii): See, e.g., Kalmbach (1983, p. 153), Example 4.

(ii) ⇒ (iii): Let $\tilde{L}$ be the MacNeille completion of L (see Theorem 1.1). We identify L with $\varphi(L)$ (where $\varphi: L \to \tilde{L}$ is an embedding). Then $\tilde{L}$ and L have the same set of all atoms. Applying Remark 2.3 and then Proposition 2.2 on L, we obtain (iii).

(iii)⇒(i) is obvious. ∎

Since a compact topological OML is complete atomic and almost orthogonal (see Theorem 1.1), from the last theorem we obtain:

Corollary 2.5. Let L be a compact topological OML. Then the following conditions are equivalent:

(i) L is modular.
(ii) L satisfies the exchange property.
(iii) L is isomorphic to a direct product of finite irreducible modular OMLs.

Remark 2.6. Every finite irreducible modular OML is of the form $\mathbf{2}^1$ or $\mathbf{MO}(n)$, $n \in N$ (Kalmbach, 1983, p. 130). This implies that every compact topological OML satisfying the exchange property is a direct product of a discrete Boolean algebra and factors of type $\mathbf{MO}(n)$.

Greechie and Herman (1985) showed that the class of commutator-finite OMLs strictly contains the class of block-finite OMLs. [Recall that an OML L is called commutator-finite (block-finite) if it contains only finitely many commutators (blocks). A block is a maximal Boolean subalgebra of L and for any $x, y \in L$ the commutator is defined by $\mathrm{com}(x, y) = (x \vee y) \wedge (x' \vee y) \wedge (x \vee y') \wedge (x' \vee y')$.] We will show that these two classes coincide for modular almost orthogonal OMLs.

Proposition 2.7. Let L be a modular, atomic, commutator-finite OML. The following statements are equivalent:

(i) L is block-finite.
(ii) L is almost orthogonal.

Proof. Evidently, every atomic Boolean algebra has both properties (i) and (ii). Assume that L is a non-Boolean, modular, commutator-finite OML. Then by Greechie and Herman (1985), Theorem 14, L has an orthogonal decomposition

$$L = [0, e_0] \times [0, e_1] \times \cdots \times [0, e_k]$$

where e_i $(0 \leq i \leq k)$ is an atom in $C(L)$, $[0, e_0]$ is a Boolean algebra, and $[0, e_i]$, $i = 1, 2, \ldots, k$, is an irreducible OML. Clearly, every $[0, e_i]$ $(1 \leq i \leq k)$ is also modular and commutator-finite, and hence in view of Greechie and

Herman (1985), Theorem 16, it is of type $\mathbf{MO}(n)$, $n \geq 2$, for some cardinal number n. Now, from the decomposition of L it follows that L is block-finite if and only if every $[0, e_i]$ $(1 \leq i \leq k)$ is finite. Hence L is block-finite if and only if L is almost orthogonal. ■

Since every block-finite OML is commutator-finite, Proposition 2.7 implies that in the class of modular atomic OMLs, any two of the three conditions block-finite, commutator-finite, and almost orthogonal imply the third one. As a corollary, we obtain the following statement.

Theorem 2.8. The class of modular almost orthogonal OMLs is the largest subclass of the class of modular atomic OMLs in which the conditions block-finite and commutator-finite are equivalent.

3. FAITHFUL VALUATIONS

A finite *valuation* on an OML L is a map $v: L \to \langle 0, \infty)$ such that $v(0) = 0$ and $v(x \vee y) = v(x) + v(y) - v(x \wedge y)$ for every $x, y \in L$. A valuation v on L is *faithful* if $v(x) = 0$ implies $x = 0$, $x \in L$. It is a well-known fact that the existence of a faithful finite valuation on an OML L implies that L is *separable* (i.e., every set of mutually orthogonal nonzero elements of L is at most countable) and modular (Sarymasakov *et al.*, 1983, p. 36). We show that in the case of almost orthogonal OML the opposite implication also is true. [Denote $x \Delta y = (x \vee y) \wedge (x' \vee y')$.]

Theorem 3.1. Let L be a separable modular almost orthogonal OML. Then there exists a faithful valuation $\tilde{v}: \tilde{L} \to \langle 0, \infty)$ on the MacNeille completion $\tilde{L}$ of L such that for any net $(\tilde{x}_\alpha)_\alpha \subseteq \tilde{L}$ and $\tilde{x} \in \tilde{L}$

$$\tilde{x}_\alpha \xrightarrow{(o)} \tilde{x} \text{ (in } \tilde{L}) \quad \text{iff} \quad \tilde{v}(\tilde{x}_\alpha \Delta \tilde{x}) \to 0$$

The restriction $v = \tilde{v}/L$ of $\tilde{v}$ to L is a finite faithful valuation on L such that for any net $(x_\alpha)_\alpha \subseteq L$ and $x \in L$

$$x_\alpha \xrightarrow{\tau_0} x \text{ (in } L) \quad \text{iff} \quad v(x_\alpha \Delta x) \to 0$$

Proof. In view of Theorem 2.4 and Proposition 2.2, the MacNeille completion of L is $\tilde{L} = \prod_{i \in I} [0, c_i]$, where for every $i \in I$, $[0, c_i]$ is a finite OML and $c_i \in C(\tilde{L}) \cap L$. The separability of L implies that the set I is at most countable. Without loss of generality we can assume that I is infinite. For $i = 1, 2, \ldots,$ there exists a faithful valuation $v_i: [0, c_i] \to [0, 1/2^i]$. For $x = (x_i)_{i=1}^{\infty} \in \tilde{L}$, let us define $\tilde{v}(x) = \sum_{i=1}^{\infty} v_i(x_i)$. It is easy to prove that $\tilde{v}$ is an

(o)-continuous faithful valuation on $\tilde{L}$ with $\tilde{v}(1)=1$. Indeed, let $\tilde{x}$, $\tilde{y}\in\tilde{L}$. Then

$$\begin{aligned}\tilde{v}(\tilde{x}\vee\tilde{y}) &= \sum_{i=1}^{\infty} v_i((\tilde{x}\vee\tilde{y})\wedge c_i)\\ &= \sum_{i=1}^{\infty} v_i((\tilde{x}\wedge c_i)\vee(\tilde{y}\wedge c_i))\\ &= \sum_{i=1}^{\infty} (v_i(\tilde{x}\wedge c_i)+v_i(\tilde{y}\wedge c_i))-(v_i(\tilde{x}\wedge\tilde{y}\wedge c_i))\\ &= \tilde{v}(\tilde{x})+\tilde{v}(\tilde{y})-\tilde{v}(\tilde{x}\wedge\tilde{y})\end{aligned}$$

which proves that $\tilde{v}$ is a valuation. Moreover, $\tilde{v}(1)=\sum_{i=1}^{\infty} v_i(c_i)=\sum_{i=1}^{\infty}(1/2^i)=1$. Clearly, $\tilde{v}$ is faithful.

Now assume that $\tilde{z}_\alpha \nearrow \tilde{x}$, with $\tilde{z}_\alpha$, $\tilde{x}\in\tilde{L}$. Then for any $i=1,2,\ldots$, $\tilde{z}_\alpha\wedge c_i\nearrow\tilde{x}\wedge c_i$ [because $(o)=\tau$ on $\tilde{L}$]. As $[0, c_i]$ is finite, for every $i=1,2,\ldots$ there exist α_i such that for all $\alpha\geq\alpha_i$ we have $\tilde{z}_\alpha\wedge c_i=\tilde{x}\wedge c_i$. Let $\varepsilon>0$ be arbitrary. As $\tilde{v}(\tilde{x})=\sum_{i=1}^{\infty} v_i(\tilde{x}\wedge c_i)$, there exists $n_0\in\mathbb{N}$ such that $\sum_{i=1}^{n_0} v_i(\tilde{x}\wedge c_i)>\tilde{v}(\tilde{x})-\varepsilon$.

Let $\alpha_0\geq\alpha_1,\alpha_2,\ldots,\alpha_{n_0}$. Then for all $\alpha\geq\alpha_0$ we have $\tilde{z}_\alpha\wedge c_i=\tilde{x}\wedge c_i$, $i=1,2,\ldots,n_0$, and hence

$$\tilde{v}(\tilde{z}_{\alpha_0})\geq\sum_{i=1}^{n_0} v_i(\tilde{z}_{\alpha_0}\wedge c_i)=\sum_{i=1}^{n_0} v_i(\tilde{x}\wedge c_i)>\tilde{v}(x)-\varepsilon$$

On the other hand, $\tilde{v}(\tilde{z}_\alpha)\leq\tilde{v}(x)$ for all α. Therefore $\tilde{v}(\tilde{z}_\alpha)\nearrow\tilde{v}(x)$, which proves that $\tilde{v}$ is (o)-continuous.

Let $(\tilde{x}_\alpha)_\alpha\subseteq\tilde{L}$, $\tilde{x}\in\tilde{L}$. Since $\tilde{L}$ is a direct product of finite OMLs, L is (o)-continuous and thus $\tilde{x}_\alpha\rightarrow^{(o)}\tilde{x}$ if $\tilde{x}_\alpha\,\Delta\tilde{x}\rightarrow^{(o)}0$ (Sarymsakov *et al.*, 1983, p. 76), and this implies $\tilde{v}(\tilde{x}_\alpha\,\Delta\tilde{x})\rightarrow 0$. Conversely, assume that $\tilde{v}(x_\alpha\,\Delta x)\rightarrow 0$ and let $(\tilde{x}_\beta\,\Delta\tilde{x})_\beta$ be an arbitrary subnet of $(\tilde{x}_\alpha\,\Delta\tilde{x})_\alpha$. Since $(\tilde{L},\tau)$ is compact and $\tau=(o)$, there exists a convergent subnet $(\tilde{x}_\gamma\,\Delta\tilde{x})_\gamma\rightarrow^{(o)}\tilde{y}\in\tilde{L}$. This implies $\tilde{v}(\tilde{x}_\gamma\,\Delta\tilde{x})\rightarrow\tilde{v}(\tilde{y})=0$ and since $\tilde{v}$ is faithful, we obtain $\tilde{y}=0$. Thus

$$\tilde{x}_\alpha\,\Delta\tilde{x}\overset{(o)}{\rightarrow}0\ (\text{in } \tilde{L})\quad\text{iff}\quad\tilde{v}(\tilde{x}_\alpha\,\Delta x)\rightarrow 0$$

Now let $(x_\alpha)_\alpha\subseteq L$ and assume that $x_\alpha\rightarrow^{\tau_0}x$ (in L). By Riečanová (1991), Theorem 2.4 (respectively Riečanová, 1990, Theorem 3), it holds that $x_\alpha\rightarrow^{\tau_0}x$ (in L) iff $x_\alpha\rightarrow^{(o)}x$ (in $\tilde{L}$), which holds iff $\tilde{v}(x_\alpha\,\Delta x)=v(x_\alpha\,\Delta x)\rightarrow 0$.

As a consequence of this theorem and the preceding results, we obtain:

Theorem 3.2. Let L be an almost orthogonal OML. The following statements are equivalent:

(i) There exists a finite faithful valuation on L.
(ii) L is modular and the set of all atoms in L is at most countable.
(iii) There exists an (o)-continuous faithful valuation v on L such that

$$x_\alpha \xrightarrow{\tau_0} x \text{ (in } L) \quad \text{iff} \quad v(x_\alpha \,\Delta x) \to 0$$

(iv) L is separable and has the exchange property.
(v) L is residually finite and its MacNeille completion is isomorphic to a direct product of at most countably many finite irreducible modular OMLs.

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